

Non-perturbative relativistic guiding center transformation: exact magnetic moment and the gyro-phase proposed as the *Kaluza-Klein* 5^{th} dimension

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Abstract. The non perturbative guiding center transformation [Di Troia C., Phys. Plasmas **22** 042103 (2015)] is extended to the relativistic regime. The single particle dynamic is described in the Minkowski flat space-time. The main solutions are obtained in covariant form: the gyrating particle solutions and the guiding particle solution, both in gyro-kinetic as in MHD orderings. It is shown the relevance of the ideal Ohm's law in the context of the guiding center transformation. Moreover, it is also considered the presence of a gravitational field. The way to introduce the gravitational field is original and based on the Einstein conjecture on the feasibility to extend the general relativity theory to include electromagnetism. In gyro-kinetic theory, some interesting novelties appear in a natural way, such as the exactness of the conservation of magnetic moment, or the fact that the gyro-phase is treated as the non observable fifth dimension of the *Kaluza-Klein* model.

1. Introduction

In plasma physics, the gyrokinetic codes are heavily used because they offer the possibility to understand plasma mechanisms from firsts principles. The collective dynamic is the effect of the self-consistent interaction of single particles with electromagnetic fields. The particle interaction with electromagnetic (e.m.) fields is described by the *Lorentz's force law*, whilst the e.m. fields are described by Maxwell's equations. The difficulty is in the nonlinearity of the problem, because the same e.m. fields that influence the motion of the single particle are sustained by the charge four-current density made by the same particles. This problem is so difficult that some approximations must be done. Two of these approximations are often applied: the motion of the particles is approximated and the relativistic effects are neglected. In the present work we use a *non*-perturbative approach for describing the particle relativistic motion in a self-consistent e.m. field. Moreover, mainly for astrophysical and cosmic plasmas, the present description is extended to a *general relativistic* formulation when the presence of a gravitational field is

not neglected. It is worth noticing that the solution of an exact *Vlasov-Maxwell-Einstein* system gives the most complete description of what concerns the *classical* field theory approach for studying plasmas.

In section II, the non perturbative guiding center transformation [1], based on the Lorentz's force law solutions, will be shown and extended to relativistic regimes. Within such approach it will be possible to analytically describe the motion of a charged (classical) particle in a general e.m. field. Some trivial solutions are shown in section III. These are the guiding particle solution which is minimally coupled with the magnetic field and the gyrating particle solution that describes a closed orbit trajectory spinning around a fixed guiding center. Such solutions are unstable and, for this reason, not seen in Nature, nevertheless other stable solutions of the Lorentz's force law can be constructed from the combination of those trivial and unstable solutions.

The relevance of the relativistic plasma *ideal Ohm's law* will be stressed. In *magneto-hydrodynamics* (MHD), the ideal Ohm's law is used for describing the plasma flow. However, in the present analysis the relativistic plasma flow is replaced by the relativistic guiding center velocity field, which is a function of the guiding center position at a certain time.

Adopting the same lagrangian formalism used for the magnetic force lines in [2], the particle dynamic will be considered with different metric tensors: from a flat space-time geometry (M_4) to a curved extended phase-space (phase-space plus time) geometry, in section IV. When the guiding center coordinates are used, it will be possible to apply the *Kaluza-Klein* (KK) mechanism [3, 4] with a geometry $\mathbb{R}^{3,1} \times S^1$ for the extended phase space.

Some important results are obtained, *e.g.* it will be shown the exactness of the conservation of the magnetic moment, no longer an adiabatic invariant of motion, being assumed the cyclic behavior of the gyro-phase, as in the *Routhian* reduction scheme [5]. However, the most surprising novelty regards the fact that the gyro-phase comes to be the 5th dimension of the KK model. Commonly, in extended theories of general relativity, the metric tensor which gives the geometry for measuring distances is considered only for the spacetime. The human perception of three spatial dimensions and one time dimension, 3 + 1 dimensions, is replaced by increasing the dimensionality of one exotic extra dimension of space, like in the KK model, or more space extra-dimensions, as for string theory and M-theory. Kaluza, firstly, shown that a 5th spatial dimension can be used for obtaining both gravitation and electromagnetism, directly from a geometric general relativity approach: the *Kaluza miracle*. Klein, enlighten the *cylinder condition* for explaining why such extra-dimension can be non observable: it is rolled up with a radius so little not to be detectable. It is said that such extra-dimension is *compactified*. In the present work, the gyro-phase will be proposed as the 5th extra-dimension. Thus, it is not a space dimension but a variable that comes from the velocity space. There is no longer any need to invent a new dimension, what is needed is to give a geometry to the extended phase-space, which is something innovative with respect to what is commonly

done in general relativity theory. Thus, a *neo-classic* unification scheme is allowed.

2. Basic Equations

A charged particle (charge e and mass m) that moves in a *given* e.m. field is classically described by the Lorentz force:

$$\frac{d}{dt}\gamma_v v = \frac{e}{m}(E + v \times B), \quad (1)$$

for $c = 1$. The relativistic factor is $\gamma_v^{-1} = \sqrt{1 - v^2}$ in the flat Minkowski spacetime. If s is the *proper time* or the *world line coordinate*, then $\gamma_v^{-1} = \dot{s}$, where the dot is indicating the time derivative. In (1), $v = \dot{x}$ is the velocity. To obtain the solutions of (1), we use the *newtonian* idea of a deterministic world. Following [1], supposing to know the exact solutions of the motion, in such a way that it is possible to fix the velocity, v , for each point of the space (traced by the particle), x , at each time, t : $v = v(t, x)$. The motion will also depend by other quantities, *e.g.* the initial energy ε_0 , being $\varepsilon = \gamma_v + e\Phi/m$ (Φ is the electric potential), or the initial velocity, v_0 . However, we treat such variables as constant parameters and, for the moment, they are not explicitly considered. The total derivative with respect to time is:

$$\frac{d}{dt}\gamma_v v = \partial_t \gamma_v v + v \cdot \nabla \gamma_v v = \partial_t \gamma_v v + \gamma_v^{-1} \nabla \frac{\gamma_v^2 v^2}{2} - v \times \nabla \times \gamma_v v. \quad (2)$$

Introducing the e.m. potentials, Φ and A , in (1) then the equation (2) becomes

$$\partial_t(\gamma_v v + eA/m) + \gamma_v^{-1} \nabla \frac{\gamma_v^2 v^2}{2} + (e/m) \nabla \Phi = v \times [(e/m)B + \nabla \times v]. \quad (3)$$

From the identities $\gamma_v^{-1} \nabla \gamma_v^2 v^2 / 2 = \gamma_v^{-1} \nabla \gamma_v^2 / 2 = \nabla \gamma_v$, it follows:

$$\partial_t(\gamma_v v + eA/m) + \nabla(\gamma_v + e\Phi/m) = v \times \nabla \times (\gamma_v v + eA/m). \quad (4)$$

The latter equation can be suggestively read introducing the "canonical" e.m. fields $E_c = -(m/e)\nabla\varepsilon - (m/e)\partial_t p$ and $B_c = (m/e)\nabla \times p$. In fact, E_c and B_c are said "canonical" because of the potentials, $\Phi_c = (m/e)\varepsilon$ and $A_c = (m/e)p$, that are the energy and momentum, *i.e.* the canonical variables of time and position, respectively. Now, the equation (3) is rewritten as

$$E_c + v \times B_c = 0, \quad (5)$$

which means that in the reference frame that moves with the particle, $\dot{x} = v(t, x)$, there are no net force on the charge even if the e.m. fields are different from zero. This is the *free-fall reference frame* for electromagnetism and something similar to the *equivalence principle* can also be stated here. We refer to equation (5) as the ideal *Ohm's law* because in plasma physics it is written in the same way: $E + V_p \times B = 0$, being V_p the plasma flow. The reason to give this name to an apparently different equation is that this equation gives the same solutions. Thus, even if the context is different, it is always possible to imagine a plasma with velocity $V_p = v$ in an e.m. fields (E_c, B_c) , so that the equation (5) results in being the ideal Ohm's law. An excellent recent work on the

true relativistic Ohm's law is [6], where some similarities to the present solutions can be found.

It is possible to write the solution of Ohm's law as

$$v = v_b b + \frac{E_c \times B_c}{B_c^2}, \quad (6)$$

where b is the unit vector of the canonical magnetic field, $B_c = |B_c|b$, and $E_c \times B_c/B_c^2$ is the $E \times B$ -like *drift* velocity.

In plasma physics, it is interesting to study the case corresponding to the *gyro-kinetic* ordering that neglects the $E \times B$ drift. If $E_c = 0$ then the solution is $v = v_b b$: for $\omega_t = (e/m)|B_c|/\gamma_v$, then

$$\gamma_v v = \frac{v_b}{\omega_t} \left(\frac{e}{m} B + \nabla \times \gamma_v v \right), \quad (7)$$

which is the case analyzed in [1]. Indeed, if $\lambda = v_b/\omega_t$ and $u = \gamma_v v$, then the latter becomes the *fundamental equation*:

$$\frac{u}{\lambda} = \left(\frac{e}{m} B + \nabla \times u \right), \quad (8)$$

as in the non-relativistic case [1].

Equation (8) is said fundamental because its solution gives the answer for many problems encountered in plasma physics and/or electrodynamics. At first, if $\lambda \rightarrow \infty$ then $B = -(m/e)\nabla \times u$ and the velocity becomes strictly related to the vector potential: the problem is to find a vector potential from a given magnetic field. This kind of solution will be called the *gyrating particle* solution. Secondly, If $(e/m) \rightarrow 0$ then $u = \lambda \nabla \times u$, which is recognized as the *force free equation* [7, 8] that denotes the *Beltrami field* [9]. In [1], equation (8) is treated as the non-homogeneous version of the *force free* equation. Finally, the *guiding particle solution* is obtained when the vorticity, $\nabla \times u$, is small: $\nabla \times u \sim 0$. In this case, the velocity is mostly parallel to the magnetic field $u \sim (e/m)\lambda B$. and the vorticity gives the drift velocity [1]: $v_D = \lambda \nabla \times u$. In [10], the same equation is part of a system of equations; the equation (8) is coupled with another similar equation that describes the magnetic field, and the system is used for describing interesting diamagnetic structures in plasmas.

2.1. Covariant derivation from Hamilton's principle

When relativistic energies are considered it is important to give a covariant description. In this section the spacetime is considered *Minkowskian* (flat geometry) with signature $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$. Let's start from the scalar *Lagrangian*:

$$L = -1 + (e/m)(A \cdot u - \Phi \sqrt{1 + u^2}), \quad (9)$$

being $\sqrt{1 + u^2} = \gamma_v$. We indicate with the *prime* the derivative with respect to the world line coordinate, s , so that $u = x'$ is the relativistic velocity. The lagrangian (9) is the sum of two effects, the free single particle lagrangian is $L_{\text{free}} = -1$ while

the lagrangian expressing the interaction between matter and the e.m. field is $L_{\text{ime}} = (e/m)(A \cdot u - \Phi\sqrt{1+u^2})$. It is worth noticing that with the present notation both the charge and the mass of the particle appear only in L_{ime} (the lagrangian is a real number). Adopting the *summation convention* and for $u^\alpha u_\alpha = 1$ with $\alpha = 0, 1, 2, 3$, the lagrangian can be re-written in the familiar form

$$L = -u^\alpha(u_\alpha + eA_\alpha/m), \quad (10)$$

being $A_0 = \Phi$ the electric potential. Explicitly, we have assumed that the *contravariant* velocity is $u^\alpha = (\gamma_v, \gamma_v v)$, while the *covariant* velocity is obtained from the product $u_\alpha = \eta_{\alpha\beta}u^\beta$, that gives $u_\alpha = (\gamma_v, -\gamma_v v)$. In the lagrangian (10) the corresponding differential form is sometimes known as the *Poincaré-Cartan form*: $Lds = -p_\alpha dx^\alpha$.

From $1 = 1/2 + u^\alpha u_\alpha/2$, an equivalent lagrangian can be

$$L = -\frac{u^\alpha u_\alpha}{2} - \frac{e}{m}u^\alpha A_\alpha - \frac{1}{2}. \quad (11)$$

It is worth to note that such lagrangian is very similar to the non relativistic one, $L_{\text{nr}} = v^2/2 + (e/m)(v \cdot A - \Phi)$; if $u \rightarrow v$ then the difference is only due to the energy at rest, which is absent in L_{nr} . From (11) it is now possible to obtain the canonical covariant momentum:

$$p_\alpha \equiv -\frac{\partial}{\partial u^\alpha}L = u_\alpha + (e/m)A_\alpha. \quad (12)$$

The *Euler-Lagrange* (E-L) equations:

$$\frac{d}{ds}\frac{\partial}{\partial u^\nu}L - \partial_\nu L = 0, \quad (13)$$

give the equations of motion:

$$\frac{d}{ds}p_\mu = (e/m)u^\nu \partial_\mu A_\nu. \quad (14)$$

From $p'_\alpha = u'_\alpha + (e/m)A'_\alpha = u'_\alpha + (e/m)u^\beta \partial_\beta A_\alpha$, the covariant Lorentz's force law is

$$u'^\beta = (e/m)u^\beta(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = (e/m)u^\beta F_{\alpha\beta}, \quad (15)$$

being $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$ the *Maxwell* tensor. Finally, the time component of the latter equation is

$$\frac{d}{ds}\gamma_v = \gamma_v \dot{\gamma}_v = (e/m)\gamma_v v \cdot E, \quad (16)$$

or $\dot{\gamma}_v = (e/m)v \cdot E$, that means that only the parallel velocity to E contributes to the kinetic energy variation[‡]. Moreover, $\dot{\varepsilon} = \dot{\gamma}_v + (e/m)\dot{\Phi} = (e/m)(\partial_t \Phi - v \cdot \partial_t A)$, as in the non relativistic limit.

As before, we suppose to know the solutions on the spacetime traced by the particles: $u^\alpha = u^\alpha(x^\beta(s))$ for given $F_{\alpha\beta} = F_{\alpha\beta}(x^\gamma(s))$. These solutions also depend on the initial velocities that are treated as constant parameters and, at the moment, are ignored. The derivative of u^α with respect to s can now be written as

$$u'_\alpha = u^\beta \partial_\beta u_\alpha = u^\beta \partial_\beta u_\alpha - \partial_\alpha(u^\beta u_\beta)/2 = u^\beta(\partial_\beta u_\alpha - \partial_\alpha u_\beta), \quad (17)$$

[‡] Commonly the note for the latter equation is quite incomplete because it is simply said that only the electric field, not the magnetic one, is able to change the kinetic energy of a particle.

having used the identity $u^\beta u_\beta = 1$. By substituting (17) into (15), the solution of the Lorentz's force law must satisfy also the following equation:

$$u^\beta (\partial_\alpha u_\beta + \partial_\alpha (e/m) A_\beta - \partial_\beta u_\alpha - \partial_\beta (e/m) A_\alpha) = 0. \quad (18)$$

This is important because we can recognize the canonical momentum, $p_\alpha = u_\alpha + (e/m) A_\alpha$ for defining the *canonical Maxwell* tensor, $F_{c\alpha\beta}$, and the angular frequency tensor, $\omega_{\alpha\beta}$, directly from:

$$(e/m) F_{c\alpha\beta} \equiv \omega_{\alpha\beta} \equiv \partial_\alpha p_\beta - \partial_\beta p_\alpha. \quad (19)$$

The solutions of motion must satisfy the following covariant ideal *Ohm's law*:

$$u^\beta \omega_{\alpha\beta} = 0. \quad (20)$$

Thus, equation (5) is found when $\alpha = 1, 2, 3$; whilst, if $\alpha = 0$ then

$$\gamma_v v \cdot (-\nabla \varepsilon - \partial_t p) = 0, \quad (21)$$

which means that E_c is transversal to v , so that it doesn't contribute to the energy variation (for this reason, in [1], E_c was indicated as E_t).

Even if we have already obtained the covariant equations, it is instructive to derive the same equations (20) directly from the most simple lagrangian: $L = -u^\alpha p_\alpha(x^\beta)$, which is the same of (10) but now the covariant momentum, $p_\alpha = p_\alpha(x^\beta)$, is only function of the spacetime coordinates and it doesn't depend on the (relativistic) velocity. For such lagrangian, the four momentum is

$$p_\alpha(x^\beta) \equiv -\frac{\partial}{\partial u^\alpha} L \quad (22)$$

and the E-L equations are:

$$u^\beta \partial_\beta p_\alpha = u^\beta \partial_\alpha p_\beta, \quad (23)$$

being $p'_\alpha = u^\beta \partial_\beta p_\alpha$. The former is exactly the equation (20).

3. Solutions of the relativistic ideal Ohm's law

There are some simple solutions of the equation (20). The trivial solution is $\omega_{\alpha\beta} = 0$, result to be very important. Another simple solution is $\omega_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} k^\gamma u^\delta$, where $\epsilon_{\alpha\beta\gamma\delta}$ is the *Levi-Civita* symbol ($\epsilon_{0123} = 1$) and k^γ is the *wave number* four-vector. Also this solution is trivial because the *Levi-Civita* symbol is totally anti-symmetric, so that $u^\beta \epsilon_{\alpha\beta\gamma\delta} k^\gamma u^\delta = 0$ for the symmetry $\beta \leftrightarrow \delta$.

3.1. Relativistic guiding particle solution

Let's start with the analysis of the following solution: $\omega_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} k^\gamma u^\delta$ and consider the case $k^0 = 1/\lambda$ and $k = 0$. Now, $\omega_{\alpha\beta} = u^\delta \epsilon_{\delta\alpha\beta 0}/\lambda$. Thus, only the spatial components survive:

$$\omega_{ij} = \frac{u^k}{\lambda} \epsilon_{ijk} \quad \text{with } i, j, k = 1, 2, 3. \quad (24)$$

Multiplying for ϵ^{ijl} both sides of the latter equation, and using the equivalence $\epsilon_{ijk}\epsilon^{ijl} = 2\delta_l^k$, then

$$\frac{u^l}{\lambda} = \frac{\epsilon^{ijl}}{2}\omega_{ij} = \epsilon^{ijl}\partial_i p_j, \quad (25)$$

which is the l component of

$$\frac{u}{\lambda} = \frac{e}{m}B + \nabla \times u, \quad (26)$$

re-obtaining (8). Together with (8), there is also the condition $\omega_{0i} = 0$, which means:

$$\partial_t p + \nabla \varepsilon = 0, \quad (27)$$

as it should in the gyrokinetic-like ordering ($E_c = 0$). Such equation, already studied in [1], is particularly important when $u \sim \lambda eB/m$. In this case it is better to indicate u with U and refer to it as the *guiding particle relativistic velocity*. The reason is that it describes the motion of a particle, with null magnetic moment that proceeds mostly parallel to the magnetic field with a drift velocity $\lambda \nabla \times U$. For a generic magnetic field, it is possible to obtain a perturbative solution ordered in power of λ so that the 0th order approximation is

$$U^{(0)} = \frac{e}{m}\lambda B. \quad (28)$$

The leading order approximation is

$$U^{(1)} = \lambda \frac{e}{m}B + \lambda \nabla \times U^{(0)}. \quad (29)$$

If $\lambda = (m/e)u_{\parallel}/|B|$, then the former is the familiar guiding center (relativistic) velocity (for null magnetic moment) at leading order:

$$U^{(1)} = u_{\parallel}b_{(0)} + (m/e)u_{\parallel}/|B|\nabla \times u_{\parallel}b_{(0)}, \quad (30)$$

with $B = |B|b_{(0)}$. In [1], an exact solution of (8) is obtained when the magnetic field is axisymmetric as it happens in many interesting circumstances. In such case, a common representation of B is

$$B = \nabla \psi \times \nabla \phi + F \nabla \phi, \quad (31)$$

where ϕ is the toroidal angle, ψ is the poloidal magnetic flux surface and F/R is the toroidal component of the magnetic field, ($\nabla \phi = e_{\phi}/R$ with e_{ϕ} , the unit vector in the toroidal direction, and R the radial distance from the axis). The guiding particle velocity solution of (8) in an axis symmetric magnetic field like (31) is [1]

$$U = \lambda \frac{e}{m} \nabla \mathcal{P}_{\phi} \times \nabla \phi + \frac{e}{m} (\mathcal{P}_{\phi} - \psi) \nabla \phi \quad (32)$$

with the guiding particle toroidal momentum, \mathcal{P}_{ϕ} (in magnetic flux unit), satisfying the following equation:

$$\lambda R^2 \nabla \cdot \frac{\lambda \nabla \mathcal{P}_{\phi}}{R^2} + \mathcal{P}_{\phi} = 0. \quad (33)$$

The latter equation, that can be written as an eigenvalue equation for the *Shafranov operator*, was correctly obtained but wrongly written in [1] (see [11] for details).

3.2. Non-canonical lagrangian derivation of the guiding center solution

Following the illuminating work of Cary R. J. and Littlejohn R. G. [2], it is possible to find a *lagrangian* derivation of the former guiding particle solution. The point here is to describe the lagrangian mechanics using non-canonical variables of the extended phase space (phase space plus time). The (*Maupertius*) principle of least action states that:

$$\delta W = \delta \int ds p(x) \cdot \frac{dx}{ds} = 0. \quad (34)$$

with $p(x) = u(x) + (e/m)A(x)$ and $p(x) \cdot \delta x = 0$ at the end points. The E-L equations are

$$\frac{dx}{ds} \times \nabla \times p(x) = 0, \quad (35)$$

which means that the velocity u is parallel to $\nabla \times p(x)$ or

$$u = \lambda \nabla \times p(x), \quad (36)$$

re-obtaining the fundamental equation (8). Nevertheless, written in the latter form, something unusual appears. In fact, the *Hamilton-Jacobi* solutions, that are *classical* solutions, are obtained setting $p = \nabla S$, where S is the principal *Hamilton* function§. In our case, p is not a gradient of a function, otherwise its *curl* should vanish. We have already defined the *canonical* magnetic field exactly as the *curl* of $(m/e)p$. This means that classical solutions have $B_c = 0$ and so, we are inspecting *non classical* solutions with $B_c \neq 0$.

The lagrangian $p(x) \cdot u$ is missing something; let's subtract the total derivative with respect to s of εt , being ε a constant: $\varepsilon' = 0$. In such a way that $L = p(x) \cdot u - \varepsilon \gamma_v$. It is worth noticing that we have used the property that a total derivative with respect to s can be added, or subtracted, to the initial lagrangian without changing the solutions of motion||. Now, the *Hamiltonian* is the energy multiplied for γ_v expressed as a function of coordinates and momenta: $H = p \cdot u - L = \varepsilon \gamma_v$.

Up to now, we have talked about the guiding particle as the particle satisfying (8) with null magnetic moment (and minimally coupled with the magnetic field). The same equation (8) is considered as the equation describing the *guiding center velocity* if the particle has a non vanishing magnetic moment. We can add to the lagrangian another total world line derivative without changing the equation of motion. Let's add the total derivative of the following *gauge function*:

$$g = (m/e)^2 \mu \gamma, \quad (37)$$

being μ a constant and γ an angle, that will be shown to correspond to the *exact magnetic moment* and *gyro-phase*, respectively. The new lagrangian is $\tilde{L} = L + (e/m)g' = L + (m/e)\mu\gamma'$, whilst the new hamiltonian is

$$\tilde{H} = p \cdot u - \tilde{L} = \varepsilon \gamma_v - (m/e)\mu\gamma', \quad (38)$$

§ In the present case $S = W - \varepsilon t$, where W is the *Hamilton's characteristic function* and $\nabla S = \nabla W$.

|| Under such remark it could be neglected $L_{\text{free}} = -1$ in the single particle lagrangian (9), leaving $L = L_{\text{ime}}$.

where we should express \tilde{H} as function of x and p . The world line derivative of γ , γ' , is called the *cyclotron* frequency (it is an angular frequency) and indicated by $\gamma' = \omega_c \P$. The constancy of μ can also be associated to the symmetry of the system with respect to the angle γ . It is worth to note that such symmetry is constructed when a total s derivative, which doesn't change the equations of motion, is added to the lagrangian. Indeed, if γ is considered an extra coordinate then the conjugate action is

$$J \equiv \partial_{\gamma'} \tilde{L} = (m/e)\mu, \quad (39)$$

and the *Noether* theorem says that $\mu' = 0$ because $\partial_{\gamma'} \tilde{L} = 0$. The equations of motion, once we consider γ as an extra dimension, are slightly different, depending on μ . The reason is that introducing such extra-dimension, together with its conjugated coordinate, what is described is not the motion of the particle anymore, but the motion of the origin of the reference frame where the particle motion is efficiently described, as it will be shown just below.

Starting by showing that \tilde{L} is the *relativistic guiding center Lagrangian*. In order to avoid confusion, the coordinates describing the guiding center are written in capital letters:

$$\tilde{L} = P \cdot U + (m/e)\mu\gamma' - \varepsilon\gamma_v. \quad (40)$$

Perhaps not everybody will recognize the guiding center lagrangian because only the leading order approximation is nowadays almost considered [12, 13]. The (non relativistic) *leading order guiding center Lagrangian* is

$$L_{\text{logc}} = [v_{\parallel}b_{(0)} + (e/m)A] \cdot \dot{Q} + (m/e)\mu\dot{\gamma} - H_0, \quad (41)$$

where $v_{\parallel} = U \cdot b_{(0)}$ is the parallel to the magnetic field component of the guiding center velocity, Q is the guiding center position, A is the vector potential computed at the guiding center position, $H_0 = v_{\parallel}^2/2 + \mu|B|$ is the leading order guiding center energy without electric potential (in gyrokinetic ordering). The correspondence between the most general (40) and the leading order approximation (41) is evident only if the motion of the particle is mostly parallel to the magnetic field, $U \sim v_{\parallel}b_{(0)}$ which is the 0^{th} order approximation, and if $\omega_c \sim (e/m)|B|$, for non relativistic velocities. Now, it is clear why in equation (37), μ is the *magnetic moment* and γ is the *gyro-phase*. In fact, at the 0^{th} order (uniform and constant magnetic field) the magnetic moment reduces to $\mu = v_{\perp}^2/(2|B|)$ and γ to the angle tracing the particle in a circular ring. In gyrokinetics, $P \approx v_{\parallel}b_{(0)} + (e/m)A$ is known as $(e/m)A^*$, λ^{-1} in (8) becomes $(e/m)B^*$, and ω_c is $(e/m)|B|$, when $\gamma_v \sim 1$.

However, we have introduced the concept of guiding center without giving to it a definition. Firstly, the guiding center coordinates in the presence of a magnetic field, similarly to the center of mass coordinates in a gravitational field, describe the origin of the reference frame where efficiently positions, velocities and time are measured, so

\P This is not the standard definition and it could be more appropriate to call $\gamma' = \omega_{ab}$, the *deBroglie* angular frequency because it transforms like an energy.

that

$$\begin{aligned} x &= Q + \rho(\gamma) \\ u &= U + \nu(\gamma) \\ t &= t_b + \tau(\gamma). \end{aligned}$$

It is worth noticing that the last equation is often written in plasma physics as $t = t_{\text{slow}} + t_{\text{fast}}(\gamma)$, so splitting what depends on slow variations to what depends on fast variations. In the present analysis t_b is considered as a time *before* t . It is worth to note that the guiding center transformation is simply a translational transformation on the extended phase space. All the coordinates are translated by a quantity depending on $\gamma \in S^1$. This property allows the following new definition to emerge: *the guiding center reference frame is the particular reference frame where the particle moves in a closed orbit with a periodic motion.* The efficiency on describing the general motion is just because the orbit is reduced to a closed loop parametrized by the angle γ . In order to reach such reference system occurs to subtract the relativistic guiding center velocity U from u and, moreover, shift the position of the particle to the guiding center position Q . In the guiding center reference frame it is possible to observe that the particle is gyrating in a closed loop with the cyclotron frequency. Such solution, called the gyrating particle solution, will be described shortly after having explained the main consequences of the lagrangian (41) in terms of *Lagrange* and *Poisson parenthesis*.

3.3. Lagrange and Poisson parenthesis

The present section is quite technical, but it is important to describe what concerns the dimensional reduction of a system. Historically, the dimensional reduction was a technique used to attack a complicated problem by progressively reducing it in order to reach a resolvable system. In gyro-kinetic the dynamic of the particle is separated from the fast gyro-motion reducing the analysis to the dynamic of the guiding center (if fluctuations are turned off). In the KK mechanism [3, 4], the same particle dynamic, now extended to consider also the presence of a gravitational field, is reduced from a five-dimensional to a four-dimensional space-time, leaving the 5th dimension unobservable. Thus, the *Routhian* reduction scheme [5] is a method implemented to describe a mechanical system where the reduction is made to suppress an angle coordinate after a smart change of variables. We will see how all these reduction schemes can be seen as different approaches for disregarding the gyro-phase from the equations of motion. However, in the present section we want to show why it is possible to reduce the dimensionality of a system cutting out a coordinate from the description of motion.

The 1-form associated to the guiding center *lagrangian* is

$$\tilde{L}d\tilde{s} = PdQ + (m/e)\mu d\gamma - \varepsilon dt. \quad (42)$$

For such system the motion is described by the variables $z^a = (t, Q, \gamma)$, with index a from 0 to 4, so that the world line coordinate, \tilde{s} , is function of z^a : $\tilde{s} = \tilde{s}(z^a)$. Moreover,

the conjugate momenta, w_a , are easily introduced consistently with the lagrangian in (42): $w_a = (\varepsilon, -P, -(m/e)\mu)$. Now,

$$\tilde{L} = -w_a z'^a, \quad \text{for} \quad a = 0, 1, 2, 3, 4. \quad (43)$$

However, following the analysis done in [2], it is also possible to extend the description of motion to the whole extended phase space. It is useful to consider the lagrangian in (42) as the reduced lagrangian of the entire lagrangian that operates on the extended phase space, \hat{L} . Now the indexes, A,B, ..., go from 0 to 6 and the generalized coordinate is $z^A = (t, Q, \gamma, \varepsilon, \mu)$. It is worth noticing that we are adopting non-canonical coordinates. Here, we will refer to $z^A = (t, Q, \gamma, \varepsilon, \mu)$ as the guiding center coordinates. As before, it is possible to associate a set of conjugate momenta to such variables. The new co-momenta are $w_A = (\varepsilon, -P, -(m/e)\mu, 0, 0)$, as similarly chosen in [2] for a different problem. Thus, the lagrangian can be written as

$$\hat{L} d\hat{s} = -w_A dz^A, \quad \text{for} \quad A = 0, 1, 2, 3, 4, 5, 6. \quad (44)$$

The scalar character of the lagrangians, (43) like (44), is always preserved and it is possible to change coordinates from $z^A \rightarrow Z^A$ and $w_A \rightarrow W_A$ leaving unaltered the *Poincaré-Cartan form*: $-w_A dz^A = -W_A dZ^A$. This means that the *principle of relativity* can be generalized to the extended phase space: a change of coordinates of the extended phase space preserves the physics.

The E-L equations for (44) are

$$\omega_{AB} \frac{dz^B}{d\hat{s}} = 0, \quad (45)$$

with

$$\omega_{AB} = \partial_A w_B - \partial_B w_A. \quad (46)$$

We have just shown that it is possible to find what it can be called the *ideal Ohm's law* (compare (20) with (45)) in 7 dimensions (or 5 dimensions if the motion is described through ε and μ , because $\varepsilon' = \mu' = 0$). The generalized angular frequency tensor is ω_{AB} , whose mathematical meaning is below illustrated.

If we split the 0^{th} -component, which refers to the time, from the rest, then

$$\omega_{AB} = \frac{\partial}{\partial z^A} w_B - \frac{\partial}{\partial z^B} w_A = \partial_A w_B - \partial_B w_A, \quad \text{for} \quad A, B = 1, 2, 3, 4, 5, 6. \quad (47)$$

From (45),

$$\omega_{AB} \dot{z}^B + \partial_A \varepsilon - \partial_t w_A = 0, \quad (48)$$

where we have replaced \hat{s} with t , because (45) is independent from the relativistic factor $d\hat{s}/dt$. Explicitly, ω_{AB} is the Lagrange parenthesis:

$$\omega_{AB} = \partial_A w_C \delta_B^C - \partial_B w_C \delta_A^C = \partial_A w_C \partial_B z^C - \partial_B w_C \partial_A z^C \equiv [z^A, z^B], \quad (49)$$

for $A, B, C = 1, 2, 3, 4, 5, 6$. For these reason, ω_{AB} is called the *Lagrange tensor*. Two properties of the Lagrange tensor must be reminded: it is totally antisymmetric and it is invariant to a gauge transformation: $w_A \rightarrow w_A + \partial_A g$ doesn't make any changes in

the Lagrange tensor.

If we adopt the guiding center description, we have already computed $\omega_{\alpha\beta}$ in (19), in particular

$$\omega_{ij} = [Q^i, Q^j] = \epsilon_{ijk}(e/m)B_c^k \quad \text{for } i, j = 1, 2, 3. \quad (50)$$

The other components are

$$\omega_{4j} = [\gamma, Q^j] = -\partial_\gamma P_j + (m/e)\partial_j \mu = 0 \quad \text{for } j = 1, 2, 3, \quad (51)$$

$$\omega_{45} = [\gamma, \varepsilon] = (m/e)\partial_\varepsilon \mu = 0, \quad (52)$$

$$\omega_{46} = (e/m)[\gamma, \mu] = \partial_\mu \mu = 1, \quad (53)$$

$$\omega_{56} = (e/m)[\varepsilon, \mu] = (e/m)\partial_\mu \varepsilon = 0, \quad (54)$$

$$\omega_{i5} = [Q^i, \varepsilon] = -\partial_i \varepsilon + \partial_\varepsilon P^i = u^i/|u|^2 \quad \text{for } i = 1, 2, 3, \quad (55)$$

$$\omega_{i6} = (e/m)[Q^i, \mu] = -(m/e)\partial_i \mu + (e/m)\partial_\mu P^i = 0 \quad \text{for } i = 1, 2, 3. \quad (56)$$

It is worth noticing that all the lagrange parenthesis involving γ and μ are null, $[\gamma, \mu]$ apart, which is equal to m/e . This is the reason that allows reducing the particle motion ignoring the *gyro-phase* coordinate, γ , which is said *cyclic*.

The inverse of the Lagrange tensor, if it exists, is called the Poisson tensor, whose components are expressed by the Poisson parenthesis:

$$J^{AC}\omega_{CB} \equiv \{z^A, z^C\}\omega_{CB} = \delta_B^A, \quad \text{for } A, B, C = 1, 2, 3, 4, 5, 6. \quad (57)$$

then the solution (48) is re-written as

$$\dot{z}^A = J^{AB}(-\partial_B \varepsilon + \partial_t w_B), \quad \text{for } A, B = 1, 2, 3, 4, 5, 6. \quad (58)$$

If we use canonical coordinates then the latter equations are the *Hamilton's* equations [2].

In the case of a canonical description, *e.g.* for describing orbit theory in fusion plasmas [11, 14], we can consider the symmetry in the toroidal coordinate ϕ and the conjugate invariant of motion, which is the canonical toroidal momentum, denoted by \mathcal{P}_ϕ . It is possible to define the new coordinates as Z^A , as $(t, \varepsilon, (m/e)\phi, \mathcal{P}_\phi, (e/m)\gamma, \mu, \psi)$, where ψ is the poloidal magnetic flux defined in (31); and the new momenta W_A as $(\varepsilon, 0, -\mathcal{P}_\phi, 0, -\mu, 0, -\mathcal{P}_\psi)$, where \mathcal{P}_ψ is the conjugate momentum of ψ . In such case the Lagrange tensor is

$$\Omega_{AB} = \partial_A W_B - \partial_B W_A = [Z^A, Z^B], \quad \text{for } A, B = 0, 1, 2, 3, 4, 5, \quad (59)$$

which is always null apart for the elements: $\Omega_{10} = \Omega_{23} = \Omega_{45} = 1$ and $\Omega_{01} = \Omega_{32} = \Omega_{54} = -1$. It is worth to note that the Lagrange tensor is made of *symplectic* 2×2 blocks.

The Poisson tensor has elements that can be written with the Poisson parenthesis:

$$\{Z^A, Z^B\} = \frac{\partial Z^A}{\partial Z^C} \frac{\partial Z^B}{\partial W_C} - \frac{\partial Z^B}{\partial Z^C} \frac{\partial Z^A}{\partial W_C} \quad \text{for } A, B, C = 0, 1, 2, 3, 4, 5, \quad (60)$$

and the equation of motion becomes

$$\Omega_{AB} \frac{dZ^B}{d\psi} + \partial_A \mathcal{P}_\psi - \partial_\psi W_A = 0. \quad \text{for } A, B = 0, 1, 2, 3, 4, 5. \quad (61)$$

Once it is set $\partial_\psi W_A = 0$, because ε, μ and \mathcal{P}_ψ are constants of motion, then

$$\frac{dZ^A}{d\psi} = -\{Z^A, Z^B\} \partial_B \mathcal{P}_\psi = \{\mathcal{P}_\psi, Z^A\}, \quad \text{for } A, B = 0, 1, 2, 3, 4, 5, \quad (62)$$

that are the *Hamilton's* equations with ψ as time coordinate and \mathcal{P}_ψ as *Hamiltonian*.

3.4. Ohm's solution in MHD-like orderings

Previously, we have analyzed the following solution of *Ohm's* law: $\omega_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} k^\gamma u^\delta$, with k^γ the *time-like* four-vector: $k^0 = 1/\lambda$ and $k = 0$. We have noticed that from this choice it follows that $E_c = 0$, which is said the gyro-kinetic-like ordering. Now we want to consider the case where k^γ is the *space-like* four vector $(0, k)$. In such case, $\omega_{\alpha\beta} = \epsilon_{\alpha\beta i\delta} k^i u^\delta = \epsilon_{\alpha\beta i0} k^i u^0 + \epsilon_{\alpha\beta ij} k^i u^j$. The component of $\omega_{\alpha\beta}$ are

$$\omega_{0k} = \epsilon_{0kij} k^i u^j = (k \times u)_k \quad (63)$$

and

$$\omega_{kj} = \epsilon_{kji0} k^i u^0. \quad (64)$$

That can be written in vectorial form as

$$E_c = (m/e) \gamma_v k \times v \quad (65)$$

and

$$B_c = (m/e) \gamma_v k, \quad (66)$$

being $\omega_{\alpha\beta} = (e/m) \mathcal{F}_{c\alpha\beta}$. In such case it is the wave number, and not v , that is parallel to B_c . The solution for v is the same of (6) but contrary to before the particle motion don't follow trajectories close to the magnetic field lines because of the presence of the electric field E_c . This is what happens in the MHD-like ordering. Thus, we can easily distinguish the MHD-like from the gyrokinetic ordering giving to k^σ the character of a space-like or time-like four-vector, respectively.

3.5. Gyrating particle solution

The most trivial solution of (20) is $\omega_{\alpha\beta} = 0$. In such case the canonical fields are null: $\mathcal{F}_{c\alpha\beta} = 0$, or $E_c = B_c = 0$. This is the most important solution.

If $B_c = 0$, then $(e/m)B + \nabla \times u = 0$. Now, it is possible to choose a very particular vector potential: $A = -(m/e)u + \nabla g$, being g a *gauge* function. Moreover, the function g is also seen to be proportional to the *principal Hamilton's function*, S , which is an *action*. Indeed, the canonical momentum is $p = u + (e/m)A = (e/m)\nabla g$. If $g = (m/e)S$, then $p = \nabla S$ and we have just set the initial condition for finding the *classical* solution in the *Hamilton-Jacobi* method. This is also consistent with $E_c = 0$, that means that

$\nabla u^0 + (e/m)\nabla\Phi + (e/m)\partial_t\nabla g = 0$. If $\varepsilon = u^0 + (e/m)\Phi = -(e/m)\partial_t g$, then we found the other *Hamilton-Jacobi* equation: $\varepsilon + \partial_t S = 0$, being $S = (e/m)g$.

Whilst describing these solutions, it is possible to take the following representation for the magnetic field: $B = \nabla\Psi \times \nabla\gamma$, which is commonly called *Clebsh* representation. Ψ and γ are said *Clebsh potentials*. Topologically, it is possible to choose $\gamma \in S^1$, in such a way that, in this case, it is considered the *gyro-phase*. The variable Ψ is the magnetic flux linked to the closed loop traced by γ . $\nabla\Psi$ is orthogonal to $\nabla\gamma$, in such a way that B doesn't depend on γ . The particle, that travels along the closed loop of curvilinear coordinate γ , always feels the same orthogonal magnetic force. This happens for the particular representation of the magnetic field, not because the magnetic field is straight and uniform. For this reason, such representation is also known as the straight field line representation. The motion of the gyrating particle is expressed by:

$$\begin{aligned} x &= Q + \rho(\gamma) \\ u &= U + \nu(\gamma) = \nu(\gamma), \end{aligned}$$

where we have set $U = 0$, so that the guiding center, Q , is fixed. The gyrating loop motion is well described if we set

$$\nu = \rho \times \Omega, \tag{67}$$

where Ω is the relativistic angular frequency, that depends on the position of the particle. It is possible to choose a local term of orthogonal unit vectors: $e_\gamma \cdot e_\rho \times b = 1$, where $\rho = |\rho|e_\rho$, $\Omega = \omega_c b$, with $\omega_c = \omega_c(Q, |\rho|)$, and $\nu = |\rho|\omega_c e_\gamma$.

Such description is also valid for the non relativistic case [1]. It is worth noticing that $|\rho|$ becomes a conserved quantity. In fact, the world line derivative of $\rho^2/2$ is $\rho \cdot \rho' = \rho \cdot \rho \times \Omega = 0$. Thus, $|\rho|$ doesn't depend on γ . However, $|\rho|$ depends on the magnetic flux linked to the closed orbit. In other words distances are now measured in terms of Ψ . Moreover, also the time of one revolution depends on Ψ , so that it can be considered a good time-like coordinate.

Until now, we have described a closed trajectory on the surface of a sphere, S^2 , of radius $|\rho|$. However, if other coordinates are used then the same particle is seen to move on a helicoidal trajectory. The circle S^1 is both the representation of the particle orbit, but also the description of the gyrating motion in the guiding center coordinates. What is important is that there is no difference from the point of sight of the particle. The particle feels to move in a circle, ignoring the rest of the world because can only feel the Lorentz force with the same magnetic field intensity, which is always orthogonal to its direction of motion when it is in the guiding center reference frame. In a certain sense, it is similar for a massive body in a gravitational field: the massive body moves straight along the geodesic but the spacetime is curved due to the presence of a gravitational field and the body is seen from an observer, *e.g.* to fall versus another massive body. On the same footing, a charged particle moves circularly but the spacetime is measured in units of magnetic field and if such magnetic field is non uniform then the charge is seen from an observer with a relative velocity $-U$, *e.g.* to follow a helicoidal trajectory.

In the next section, it will be shown how the electromagnetism can be described within the formalism of general relativity.

4. Kaluza-Klein solution

The coordinates z^A with $A = 0, 1, 2, 3, 4, 5, 6$, introduced in the previous section, belongs to the extended phase space. As for general relativity, where it is given a geometry to the space-time, in this section a geometry is given to the extended phase-space.

We have seen that in the presence of e.m. fields, it is useful to describe the motion in guiding center coordinates, $z^A = (t, Q, \gamma, \varepsilon, \mu)$. For accuracy, the guiding center transformation is the map, \mathcal{T} , that allows to describe particles through the guiding center coordinates, $\mathcal{T} : (t, x, p) \rightarrow (t, Q, \gamma, \varepsilon, \mu)$. It is worth noticing that the vector Q indicates the position of the guiding center, not of the particle. If $\mu \neq 0$ then the particle is elsewhere from Q .

The KK mechanism was used in the past to explain the presence of gravitation and electromagnetism thanks to the addition of, at least, a new coordinate of spacetime. The KK model can be obtained from a *Hilbert-Einstein* (HE) action extended to a space-time of five dimensions. However, in the present approach, we adopt the same mechanism, in which the new dimension is a coordinate that belongs to the velocity space. In fact, the 5th dimension is identified with the gyro-phase coordinate, γ . As a consequence we are changing the paradigm of the general relativity theory that takes into account only the space-time geometry. Thus, if you want to describe gravity then you can consider only the geometry of space-time, whilst if you want to describe gravity plus electromagnetism you have to consider the geometry of the extended phase space. Mathematically, it is not so difficult to extend the general relativity formalism to five or more (seven) dimensions. However, the physical interpretation of an *Einstein equation* in extended phase space, is quite unusual to be exposed in the present work. What is proposed here is a minimal change of the KK model and the use of the relativistic guiding center transformation. In this section we leave the *Minkowski* metric for a pseudo-*Riemannian* one.

Let's start from the *Poincaré-Cartan* one-form in (43): $\hat{L}d\hat{s} = -w_A dz^A$, for $A = 0, 1, 2, 3, 4, 5, 6$. The same one-form can be written as

$$\hat{L}d\hat{s} = -\hat{g}_{AB}w^B dz^A, \quad (68)$$

being \hat{L} a scalar quantity and where \hat{g}_{AB} is the metric tensor with the property that $w_A \equiv \hat{g}_{AB}w^B$. Thus, w^B are the *contravariant* momenta. Once the metric tensor appeared, it is possible to apply a variational principle for finding it. For this reason, it is considered a *distribution lagrangian* where the single particle lagrangian is multiplied for the distribution of masses and, then, summed to the HE lagrangian in extended dimensions. In the following distribution lagrangian,

$$\ell a = f_m \hat{L} - \frac{\hat{\mathcal{R}}}{16\pi\hat{G}}, \quad (69)$$

f_m is the scalar distribution function of masses, \hat{G} and $\hat{\mathcal{R}}$ are the gravitational constant and the *scalar curvature* for the extended phase space, respectively. The scalar curvature is defined as usual:

$$\hat{\mathcal{R}} = \hat{g}_{AB} \hat{Ric}^{AB}, \quad (70)$$

again, \hat{Ric}^{AB} is the *Ricci tensor* in the extended phase space which is furnished of a *Levi-Civita connection*. The lagrangian, (69), is a *lagrangian distribution* because the action is found when integrating ℓa over the extended phase space. If $\sqrt{-\hat{g}}$ indicates the square root of minus the determinant of the extended phase space metric, then the extended phase space volume element, $d\mathcal{M}$, can be written as:

$$d\mathcal{M} = \sqrt{-\hat{g}} d^7z, \quad (71)$$

if the guiding center coordinates are used then $d^7z = dt d^3Q d\gamma d\varepsilon d\mu$. Explicitly, the action is:

$$S = \int \ell a d\mathcal{M}, \quad (72)$$

which is a definite integration in a domain $\partial\mathcal{M}$ of the extended phase space. It is possible to separate in ℓa the effects of different contributions. A *matter lagrangian distribution*:

$$\ell a_m = -f_m, \quad (73)$$

a *field lagrangian distribution*:

$$\ell a_f = -\frac{\hat{\mathcal{R}}}{16\pi\hat{G}}, \quad (74)$$

and an *interaction dynamics lagrangian distribution*:

$$\ell a_{id} = f_m(1 + \hat{L}), \quad (75)$$

The distribution of masses, f_m is taken as a scalar function: $f_m = f_m(z^A)^+$.

Within the guiding center description, f_m indicates the presence of a particle of mass m with guiding center coordinates $(t, Q, \gamma, \varepsilon, \mu)$. The particle described by f_m must be counted only once to obtain the total mass, M , of the system. The following equivalence chain of integrations is assumed for the *matter action*, S_m :

$$S_m = - \int f_m \sqrt{-\hat{g}} d^7z = - \int \rho_m \sqrt{-g} dt d^3Q = - \int M d\hat{s}, \quad (76)$$

where ρ_m is the mass density and, above all, $\sqrt{-g}$ is the square root of minus the determinant of the space-time metric. In fact, if you call $J_{\mathcal{P}}$ the quantity $\sqrt{-\hat{g}}/\sqrt{-g}$, then:

$$\rho_m = \int f_m J_{\mathcal{P}} d\gamma d\varepsilon d\mu. \quad (77)$$

⁺ We are implicitly imposing that matter cannot be created nor destroyed.

The density of masses is obtained from the integration of the distribution of masses in the velocity space. If you introduce unspecified velocities or momenta, \mathcal{P} , with the only property that allows to write the latter velocity space volume element:

$$d^3\mathcal{P} = J_{\mathcal{P}} d\gamma d\varepsilon d\mu, \quad (78)$$

then the former integral is written in the usual form:

$$\rho_m = \int f_m d^3\mathcal{P}. \quad (79)$$

Concerning the *fields action*, S_f , we wish to have:

$$S_f = - \int \frac{\hat{\mathcal{R}}}{16\pi\hat{G}} \sqrt{-\hat{g}} d^7z = - \int \frac{F_{\alpha\beta} F^{\alpha\beta}}{4} \sqrt{-g} dt d^3Q - \int \frac{R}{16\pi G} \sqrt{-g} dt d^3Q, \quad (80)$$

In order to obtain the latter result we will use the KK mechanism. However, before doing that, we are interested in studying the *interaction dynamics action* S_{id} , that should be expressed by:

$$S_{id} = \int f_m (1 + \hat{L}) \sqrt{-\hat{g}} d^7z = \int A_\mu J^\mu \sqrt{-g} dt d^3Q, \quad (81)$$

where J^α is the charge four-current density which is a field depending on (t, Q) . The former equation will be obtained in the forthcoming subsection. It is worth noticing that, if the above equations for la_m , la_f and for la_{id} , defined in (76), (80) and (81), respectively, are considered, once la is integrated in the velocity space, the lagrangian density, \mathcal{L} appears:

$$\mathcal{L} = -\rho_m + A_\alpha J^\alpha - \frac{F_{\alpha\beta} F^{\alpha\beta}}{4} - \frac{R}{16\pi G}. \quad (82)$$

The latter is exactly the lagrangian density used for describing the presence of (e.m. interacting) matter as source of a gravitational field, which gives the *Einstein* equation, together with a charge 4-current density as source of an e.m. field, which gives the *Maxwell* equations.

4.1. The misleading symmetry

In *lagrangian mechanics* the symmetries of a system are expressed by the invariance of the lagrangian under the considered transformations. In general relativity, the conservation of the energy-momentum tensor, $T^{\alpha\beta}$, is fundamental. The conservation of $T^{\alpha\beta}$ is due to the symmetry of the lagrangian under the spacetime translation: $Q^\alpha \rightarrow x^\alpha = Q^\alpha + \rho^\alpha$. This is also true if we explicitly take, $Q^\alpha = (t_b, Q)$ and $\rho^\alpha = (\tau, \rho)$; so that, $x = Q + \rho$ and $t = t_b + \tau$.

For our needs, the single particle lagrangian, $L = p \cdot u - \varepsilon \gamma_v$ can be written with a null magnetic moment term: $L = p \cdot u - \varepsilon \gamma_v + (m/e) \mu_0 \omega_{c0}$, if $\mu_0 = 0$. Now, the guiding center transformation leaves unaltered the form of the lagrangian. In the non perturbative guiding center transformation, the momentum of the particle, $p \rightarrow P$, becomes the guiding center momentum computed at the guiding center Q and at the time t , the particle relativistic velocity, $u \rightarrow U$, becomes the relativistic guiding center

velocity U . Moreover, the null magnetic moment $\mu_0 \rightarrow \mu$ becomes a positive magnetic moment so that the gyro-phase γ becomes meaningful (if $\mu = 0$ then γ is singular). The single particle lagrangian under such transformation, is

$$L = p \cdot u - \varepsilon \gamma_v = p \cdot u - \varepsilon \gamma_v + (m/e)\mu_0 \omega_{c0} \rightarrow \hat{L} = P \cdot U - \varepsilon \gamma_v + (m/e)\mu \omega_c, \quad (83)$$

which is the *guiding center Lagrangian*, already seen. The symmetry that leaves the form of L identical to the one of \hat{L} , becomes apparent if

$$(m/e)\mu \omega_c = \varepsilon \gamma_v - \mathcal{E} \gamma_v, \quad (84)$$

so that $\hat{L} = P \cdot U - \mathcal{E} \gamma_v$.

The meaning of the guiding center transformation is to reduce the dimensionality of the system discarding the gyrating motion of the particle in such a way that it is possible to describe the guiding center motion without the gyro-phase variable, γ . This is exactly what we have implicitly done in (84). To make such property more explicit we extend to relativistic energies the analysis already done in [1]. Starting from the guiding center transformation:

$$x = Q + \rho(\gamma)$$

$$u = U + \nu(\gamma)$$

$$t = t_b + \tau(\gamma),$$

it is possible to compute the quantity $\varepsilon \gamma_v = \gamma_v^2 + (e/m)\gamma_v \Phi(t, x)$ with $\gamma_v = \gamma_v(t, x, p)$:

$$\varepsilon \gamma_v = 1 + U^2 + (2\nu \cdot U + \nu^2) + (e/m)\sqrt{1 + U^2}\Phi(t_b, Q) + (e/m)\delta_\rho(\gamma_v \Phi), \quad (85)$$

with $(e/m)\delta_\rho(\gamma_v \Phi) = (e/m)[\sqrt{1 + u^2}\Phi(t, x) - \sqrt{1 + U^2}\Phi(t_b, Q)]$. Now the unwanted dependency on γ is contained in the third, the fourth and in the last term on the right. To suppress such dependency, as done in the *routhian reduction scheme* [5], these terms are replaced by the product of an action, $(m/e)\mu$, times the frequency, $\gamma' = \omega_c$:

$$(m/e)\mu \omega_c \equiv 2\nu \cdot U + \nu^2 + (e/m)\delta_\rho(\gamma_v \Phi), \quad (86)$$

analogously of what has been done in the non relativistic treatment [1]. In such a way, the hamiltonian doesn't depend on γ anymore and the equation (84) is satisfied by definition.

We have just shown that the guiding center transformation doesn't change the form of the single particle lagrangian, meaning that there is a symmetry. Such symmetry is misleading. Indeed, suppose that you observe a helicoidal trajectory made by the motion of a charged particle in a given e.m., then such trajectory should be solution of motion. However, if you zoom on the trajectory, or if you increase the sensibility of your detector, you could discover that the simple helicoidal trajectory is made by another sub-helicoidal motion. During the first observation we have simply confused the trajectory of the particle with the trajectory of the guiding center. This happens because each particle solution can be considered a guiding center solution with null magnetic moment (at least during a finite interval of time). Such guiding center solution is representative of a family of solutions with different magnetic moment. If we don't have the sufficient

resolution to observe the gyrating motion, it is possible to consider only the guiding center motion instead of the true particle motion. Moreover, this property can be iterated (not indefinitely), so that the sub-helicoidal motion can, again, hide another subsub-helicoidal motion at a finer scale. In a certain sense, when the magnetic field differs from being constant and uniform a family of solutions enriches the extended phase space of helicoidal trajectories made by other helicoidal trajectories. It is worth noticing that realistic magnetic fields are never constant and uniform and, moreover, any realistic detector doesn't have infinite resolution. Such property, when applied with criteria, becomes the starting hypothesis in all the gyro-kinetic codes used for studying magnetic confined plasmas for controlled fusion from a kinetic perspective.

Thanks to the equation (84), it is very easy to show that the action S_{id} takes the desired form (81) when guiding center coordinates are used. In fact, $\hat{L} = P \cdot U - \mathcal{E}\gamma_V = -1 + (e/m)A \cdot U - (e/m)\Phi\gamma_V$ and

$$S_{\text{id}} = \int f_m(1 + \hat{L})\sqrt{-\hat{g}}d^7z = \frac{e}{m} \int \rho_m A_\alpha \bar{U}^\alpha \sqrt{-g} dt d^3Q. \quad (87)$$

If $A_0 = \Phi$ and $J^\alpha = (e/m)\rho_m \bar{U}^\alpha$, being

$$\rho_m \bar{U}^\alpha = \int f_m U^\alpha d^3\mathcal{P}, \quad (88)$$

with $U^\alpha = (\gamma_V, U)$, then the former is exactly the relation in (81).

We have seen that the guiding center transformation, which is a particular *local* translation in the extended phase space, is a symmetry. In analogy to what happens for the *local* translation in spacetime, the conserved quantity for the present symmetry should be called the *extended energy-momentum* tensor \hat{T}_{AB} , which is obtained from the variation of $\ell a_m + \ell a_{\text{id}} = f_m \hat{L}$ with respect to the metric tensor variation, $\delta \hat{g}^{AB}$:

$$\hat{T}_{AB} \delta \hat{g}^{AB} = -2\delta(\ell a_m + \ell a_{\text{id}}) + \hat{g}_{AB}(\ell a_m + \ell a_{\text{id}}) d\delta \hat{g}^{AB}. \quad (89)$$

Now, the *Einstein tensor* for the extended phase space is obtained from the variation of ℓa_f with respect to $\delta \hat{g}^{AB}$:

$$\hat{G}_{AB} = \hat{R}ic_{AB} - \hat{R} \hat{g}_{AB}/2, \quad (90)$$

and the *Einstein equation* can be written also for the extended phase space,

$$\hat{G}_{AB} = 8\pi \hat{G} \hat{T}_{AB}. \quad (91)$$

It is worth noticing that, if confirmed, we have just realized the *Einstein's dream* of unifying gravitation and electromagnetism from a geometrical perspective. Moreover, when extending the dimensionality from four to seven it is possible to take into account many possibilities. The abelian gauge theory will be suddenly shown to come from having chosen the gyro-phase $\gamma \in S^1$ as coordinate of the velocity space but, *non abelian gauge theories* can also be described by choosing other variables with other groupal properties. However, the possibility to definitely separate in the extended phase space what belongs from spacetime and what from velocity space must be reformulated. This is a road that needs some care and it cannot be taken now. We prefer to show

the minimal five dimensional extension of gravitation explicitly using the guiding center coordinates. Such extension is sufficient to include electromagnetism. Moreover, the present description is facilitated by the work of KK, because most of the general relativity equations that we will soon encounter, have already been studied [15].

4.2. The minimal five-dimensional theory

Instead of deriving the metric tensor from a variational approach, it is possible to settle the metric tensor directly. This can be less elegant but easier to do mostly because it has already been done. The original KK mechanism needs an extension of the dimensionality of space-time by only one dimension. Only five dimensions occur to display electromagnetism and gravitation. However, we have formulated an extension to seven, not five, dimensions of general relativity. This is too general for the present scope, but we have seen that in the single particle one-form (42) there is only the variation of five coordinates: $z^a = (t, Q, \gamma)$, with a world line coordinate $\tilde{s} = \tilde{s}(z^a)$, for $a = 0, 1, 2, 3, 4$. In this subsection we re-formulate the lagrangian distribution, (69), in five dimensions and, after adopting the KK metric tensor, we prove the equation (80), which is the last equation needed to get the wanted lagrangian density (82).

The KK mechanism is used following the review articles [3] and [4]. Many books can be consulted for the computation of the Ricci tensor and *Christoffel symbols*, but a particularly interesting note inherited with the KK mechanism is [16]. If two (canonical) constants of motion coordinates are taken into account, then the description of the dynamic of a particle in the extended phase space can be *reduced* from seven to five dimensions. For the guiding center description of motion such coordinates are the energy, ε , and the magnetic moment, μ , and we can divide the extended phase space in slices of reduced phase space with assigned ε and μ . This is allowed because the co-momenta are $w_A = (\varepsilon, -P, -(m/e)\mu, 0, 0)$, where the zeros are just indicating the use of canonical coordinates in $z^A = (t, Q, \gamma, \varepsilon, \mu)$. The one-form (44) is the same of (43) which lives in five dimensions. We have indicated with the *hat* a seven dimensional quantity, *e.g.* $\hat{L}(z^A, z'^B)$, whilst with a *tilde* a five dimensional one, *e.g.* $\tilde{L}(z^a, z'^b)$. The lagrangian in (42), $\tilde{L} = P \cdot U + (m/e)\mu\omega_c - \varepsilon\gamma_v$, is always the same but it is now written with the metric tensor \tilde{g}_{ab} :

$$\tilde{L} = -\tilde{g}_{ab}w^a z'^b, \quad \text{for } a, b=0, 1, 2, 3, 4. \quad (92)$$

Also the *lagrangian distribution*, (69) can be considered into five dimensions:

$$\ell a = f_m \tilde{L} - \frac{\tilde{R}}{16\pi\tilde{G}}, \quad (93)$$

being \tilde{R} the five dimensional scalar curvature, and \tilde{G} the five dimensional gravitational constant. In practice, $\tilde{R}/\tilde{G} = \hat{R}/\hat{G}$, as if we are considering *flat* the space described by the canonical coordinates ε and μ . It is worth noticing that although in five dimensions, all the quantities can depend also on ε and μ , *e.g.* the distribution function f_m is always the distribution of masses in the whole extended phase space and it surely depends on

ε and/or μ if it describes an equilibrium [11]. Even if the action is the same, now $\sqrt{-\tilde{g}}$ should be decomposed into $\sqrt{-\tilde{g}} = \sqrt{-g}\tilde{J}_{\mathcal{P}}$, where $\sqrt{-g}$ is the square root of minus the determinant of the metric tensor \tilde{g}_{ab} , and $\tilde{J}_{\mathcal{P}}$ is the jacobian, not specified here, for measuring the density of states for assigned ε and μ . From (72) and (71), in guiding center coordinates, the action is

$$S = \int \ell a \sqrt{-\tilde{g}} \tilde{J}_{\mathcal{P}} dt d^3Q d\gamma d\varepsilon d\mu. \quad (94)$$

Finally, we use the following KK metric tensor:

$$\tilde{g}_{ab} = \begin{vmatrix} g_{\alpha\beta} - k_G^2 A_\alpha A_\beta & k_G^2 (m/e)^2 \mu A_\alpha \\ k_G^2 (m/e)^2 \mu A_\beta & -k_G^2 (m/e)^4 \mu^2 \end{vmatrix}. \quad (95)$$

being k_G a constant that will be specified below. The latter metric tensor is used to compute the five dimensional fields lagrangian, to give the following action:

$$S_f = -\frac{1}{16\pi\tilde{G}} \int dt d^3Q \sqrt{-\tilde{g}} \left\{ R + \frac{k_G^2}{4} F_{\alpha\beta} F^{\alpha\beta} \right\} \tilde{J}_{\mathcal{P}} d\gamma d\varepsilon d\mu, \quad (96)$$

where $\sqrt{-\tilde{g}}$ is $\sqrt{-\tilde{g}} = \sqrt{-g}(m/e)^2 k_G \mu$. For obtaining the standard gravitational plus e.m. fields action, k_G must be $k_G^2 = 16\pi G$, so that

$$\tilde{G} = G \int (m/e)^2 k_G \mu \tilde{J}_{\mathcal{P}} d\gamma d\varepsilon d\mu. \quad (97)$$

Finally, the fields action is

$$S_f = - \int \sqrt{-g} dt d^3Q \frac{R}{16\pi G} - \int \sqrt{-g} dt d^3Q \frac{F_{\alpha\beta} F^{\alpha\beta}}{4}. \quad (98)$$

In this way, we have obtained the lagrangian density in (82) from the five dimensional lagrangian (93). It is worth noticing that, even if the terms in the density lagrangian (82) are the desired ones, they are referring to fields on (t, Q) where Q is the guiding center position and it doesn't indicate the position of a particle. This is an effect of the misleading symmetry. The problem is that once we have integrated the distribution lagrangian, expressed in guiding center coordinates, on the velocity space, we have lost the possibility to know where the particles effectively are. Fortunately, we already know that, at some scale, an indetermination principle should be invoked. The relation between the misleading symmetry and the quantum non-locality property should be investigated.

The KK mechanism was discarded as a possible true mechanism of Nature because it has many problems. However, all these problems are inherited to explain why the fifth dimension is unobservable [3, 4]. In the present case, this is not a problem, because the fifth dimension is *measurable*, being a physical meaningful variable. The KK mechanism can be extended to more than five dimensions, generalized to include the *cosmological constant* and, most importantly, this is shown to satisfy the *Weyl* transformation [3, 4]. For simplicity, we don't examine these interesting extensions. Moreover, the present approach, that starts from the *Lorentz' force law*, is completely *Newtonian*. Thus, another big problem, as it happens with general relativity, will be its re-formulation

within the quantum mechanical rules. However, it can be shown that, thanks to the misleading symmetry and the instability of the guiding and gyrating particle solutions, the present approach is not so in conflict with quantum mechanics but, this is another story.

5. Conclusions

The non-perturbative guiding center transformation has been extended to relativistic energies. Within the relativistic regime, the same equation (5) already seen in the non relativistic treatment [1], is re-obtained. This has been called the *ideal Ohm's law* because it is the same equation applied in a different context. Thus, the solutions of motion are studied in the light of the ideal Ohm's law. The covariant formalism has been adopted to better describe the relativistic behavior. For this reason a lagrangian approach is used for re-deriving the same equation (5) in a covariant form.

Some important solutions of the ideal Ohm's law are considered in section 3. Here, it is shown the difference between the guiding particle solution in gyrokinetic-like ordering, in MHD-like ordering, and the gyrating particle solution. All these solutions are unstable and practically identical to the non relativistic case, which have been analyzed with details in [1]. The guiding particle solution is the one described by the *fundamental equation* (8), where the magnetic moment can be different from zero. The guiding center reference frame has been finally defined in a geometrical sense as *the reference frame where the particle moves in a closed orbit with a periodic motion*. The gyro-phase, γ , is the curvilinear coordinates along such a closed loop trajectory and the magnetic moment is defined as the conjugate coordinate to γ . Thus, the dynamics have been described in the gyro-center coordinates, $z^A = (t, Q, \gamma, \varepsilon, \mu)$, within the non-canonical lagrangian mechanics developed by Cary J. R. and Littlejohn R. G. [2]. The *Lagrange* and *Poisson* parenthesis have been obtained for the non-perturbative guiding center transformation. The correspondence with the ideal Ohm's law in seven dimensions is shown, (45). Moreover, a clear and known criteria to define when a dimensionality reduction is possible, is also reminded.

Finally, a general relativity approach for describing electromagnetism using the relativistic guiding center transformation is suggested. It is shown that the formalism of non-canonical lagrangian mechanics is what is needed to extend to the general relativity formalism the presence of electromagnetic dynamics. An *Einstein's equation* (91), for the extended phase space can be settled for describing both the interactions: electromagnetism plus gravitation. Moreover, it has been proved that, for the guiding center coordinates, the relevant dynamics are five dimensional as for the original KK mechanism. The density lagrangian (82), which is used for describing both gravitation and electromagnetism, is obtained. The metric tensor has been explicitly written in (95). The gyro-phase coordinate, γ , is proposed to be the fifth KK coordinate. Thus, the extra-dimension is not a unobservable spacetime dimension but a *measurable* coordinate of the velocity space used for describing motion on the extended phase

space. If $\gamma \in S^1$, which is exactly obtained only for the *nonperturbative* guiding center transformation, an *abelian gauge theory* can be settled: electromagnetism is served on the gravitational banquet. The fact that the geometry of the velocity space must be taken into consideration also for describing the same fields acting on the particles is the novelty of the present work. This idea, which is new in general relativity theory, is quite obvious for a plasma physicist, who knows the relevance of the extended phase space for describing plasmas.

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References

- [1] Di Troia C 2015 From charge motion in general magnetic fields to the non perturbative gyrokinetic equation *Phys. Plasmas* **22** 042103.
- [2] Cary J R and Littlejohn R G 1983 Noncanonical Hamiltonian mechanics and its application to magnetic field line flow *Annals physics* **151** 1.
- [3] Bailin D and Love A 1987 Kaluza-Klein theories *Rep. Prog. Phys.* **50** 1987.
- [4] Overduin J M and Wesson P S 1997 Kaluza-Klein gravity *Phys. Rept* **283** 303.
- [5] Goldstein, Herbert 1980 Classical Mechanics (2nd ed.) San Francisco, CA: Addison Wesley. pp. 352?353.
- [6] Pegoraro F 2015 Generalised relativistic Ohm's laws, extended gauge transformations, and magnetic linking *Phys. Plasmas* **22** 112106.
- [7] Lust R and Schluter A 1954, Force-free magnetic fields *Z. Astrophys.* **34** 353.
- [8] Chandrasekhar S and Kendall P C 1957 On force-free magnetic fields *Astrophysical Journal* **126** 457.
- [9] Beltrami E 1889 Considerazioni idrodinamiche *Rendiconti del Reale Studio Lombardo, Series II* **22** 122.
- [10] Mahajan S M and Yoshida Z 1998 Double Curl Beltrami Flow: diamagnetic structures *Phys. Rev. Lett.* Vol 81, **22**, 4863.
- [11] Di Troia C 2015 Bayesian derivation of plasma equilibrium for tokamak scenarios and the associated Landau collision operator *Nuclear Fusion* **55** 123018.
- [12] Littlejohn R G 1983 Variational principles of guiding centre motion *Journal of Plasma Physics* **29**(01) 111.
- [13] Cary J R and Brizard A J 2009 Hamiltonian theory of guiding-center motion *Reviews of Modern Physics* **81** 693.
- [14] Di Troia C 2012 From the orbit theory to a guiding center parametric equilibrium distribution function *Plasma Phys. Controlled Fusion* **54** 105017
- [15] Goenner F M H 2004 On the history of unified field theories *Living Rev. Relativity* **7** 2.
- [16] Straub O W 2014 Kaluza-Klein for kids *preprint viXra.org* viXra:1406.0172v2.